On the existence of classical solutions for a two-phase flow through saturated porous media

Sabine Hengst

Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39, D – 10117 Berlin, Germany

Key words: Elliptic-parabolic differential equations, two-phase flow, porous media, mathematical analysis

Abstract

In this paper an elliptic-parabolic coupled system arising from a two-phase flow through a saturated porous medium is considered. The uniqueness and the existence of classical solutions are proved. The asymptotic behavior of solutions for large time is shown, too.

1 Introduction

We consider the following elliptic-parabolic coupled system:

$$\begin{cases} \nabla \cdot q &= 0, & \text{in Q} \\ q &= -A(v_1, ..., v_N)\nabla u - B(v_1, ..., v_N), & \\ \frac{\partial v_i}{\partial t} &+ \nabla \cdot [v_i q - D_i(v_1, ..., v_N)\nabla v_i] = 0, \qquad i = 1, .., N, & \text{in Q} \end{cases} \tag{1.1}$$

with initial-boundary data

$$\begin{cases} u_{|\Sigma_1} &= u_1, \quad q \cdot \nu_{|\Sigma_2} = 0, \\ v_{i|_{\Sigma_1 \cup \Omega_0}} &= v_{i1}, \quad [v_i q - D_i(v_1, ..., v_N)\nabla v_i] \cdot \nu_{|\Sigma_2} = 0, \end{cases} \tag{1.2}$$

where $Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$, $T > 0$, $\Sigma_1 = \Gamma_1 \times [0, T]$, $\Sigma_2 = \Gamma_2 \times [0, T]$, $\Gamma_1 \subset \partial\Omega$, $\Gamma_2 = \partial\Omega \setminus \Gamma_1$, $\Omega_0 = \Omega \times \{0\}$; ν is the outward normal to $\partial\Omega$.
The system (1.1) describes the fluid-solute-heat flow through a saturated porous medium (see [2]) where, apart from constants, u stands for the pressure, q for the flux of the fluid and v_i for the temperature or the concentration of solutes in the fluid.
Equations (1.1) with $N = 1$ (see [1,7]) also include systems governing the flow of two immiscible fluids through a porous medium. Without loss of generality we can take $N = 1$, and then problem (1.1)-(1.2) reduces to the following one:

$$(I) \quad \begin{cases} \nabla \cdot q &= 0, & \text{in Q} \\ q &= -[A(v)\nabla u + B(v)], & \\ \frac{\partial v}{\partial t} &+ \nabla \cdot [vq - D(v)\nabla v] = 0, & \text{in Q} \\ u_{|\Sigma_1} &= u_1, \quad q \cdot \nu_{|\Sigma_2} = 0, & \\ v_{|\Sigma_1 \cup \Omega_0} &= v_1, \quad [vq - D(v)\nabla v] \cdot \nu_{|\Sigma_2} = 0, & \end{cases}$$

where $A(s), B(s), D(s), u_1$ and v_1 are known functions fulfilling the following conditions:

(i) : $A(s), D(s) \in C^2(\mathbb{R})$ are positive (scalar) functions satisfying

$$\begin{aligned} 0 < A_0 &\leq A(s) \leq A_1 \\ 0 < D_0 &\leq D(s) \leq D_1 \end{aligned}$$

where A_0, A_1, D_0, D_1 are constants;

(ii) : $B(s) \in C^2(\mathbb{R})$ is an n-vector function;

(iii) : $u_1, v_1 \in C^3(Q_1)$ with $Q_1 = \Omega \times [0, \infty)$;

(iv) : $\Omega \subset \mathbb{R}^n$ is a bounded, connected open set with C^3-boundary $\partial\Omega$, and $\Gamma_1 \subset \partial\Omega$ is open and non-empty in the sense of $(n-1)$ dimensional Hausdorff measure.

In [7], the existence, uniqueness and stability of classical solutions for a problem similar to (I) are proved. However, the existence argument is restricted to the two dimensional case; for higher dimensions it is required that function A is a small perturbation of a continuous function depending only on x. In [1], a degenerate elliptic parabolic system analogous to (I) is considered. Due to the degeneration nature the existence and regularity results are established only for weak solutions. In this paper we shall show the existence, uniqueness and asymptotic behavior of classical solutions for problem (I) without restriction on dimension n.
The paper is divided as follows: In Section 2 we introduce weak solutions of problem (I) and prove the uniqueness. Then in Section 3 the existence of weak solutions is established. In Section 4 we give further regularity results implying that the weak solution is also a classical one. Finally we show in Section 5 that under proper conditions the solution converges to the unique steady solution of (I), as $t \to \infty$.

2 Weak solution and uniqueness

First, we introduce weak solutions of problem (I). By a weak solution of problem (I) we mean a pair of functions (u, v) satisfying

(i) : $u \in u_1 + L^2(0, T; V),\ v \in v_1 + L^2(0, T; V) \cap L^\infty(Q)$,
where $V = \{w \in H^1(\Omega) : w_{|\Gamma_1} = 0\}$;

(ii) : for any $\varphi \in C^1(\bar{Q})$ vanishing on $\Sigma_1 \cup (\Omega \times \{T\})$ we have

$$\int_{\Omega\times\{t\}} [A(v)\nabla u + B(v)] \cdot \nabla\varphi = 0 \quad \text{a.e.} \quad t \in (0, T),$$

$$\iint_Q \left\{ v\frac{\partial\varphi}{\partial t} - [D(v)\nabla v + v(A(v)\nabla u + B(v))] \cdot \nabla\varphi \right\} + \int_{\Omega\times\{0\}} v_1\varphi = 0.$$

With some additional assumptions we get the uniqueness of the weak solutions.

Theorem 2.1: *Let (u,v) be a weak solution with $\nabla u, \nabla v \in L^\infty(Q)$. Then it is the unique weak solution of problem (I).*

Proof: Suppose that $(\tilde{u},\tilde{v})$ is another weak solution. Then for any $\varphi \in C^1(\bar{Q})$ vanishing on Σ_1 and $\Omega \times \{T\}$ we have

$$\int_{\Omega\times\{t\}} [A(v)\nabla u - A(\tilde{v})\nabla\tilde{u} + B(v) - B(\tilde{v})] \cdot \nabla\varphi = 0 \quad \text{a.e. } t \in (0,T), \tag{2.1}$$

$$\begin{aligned} &\iint_Q \{(v-\tilde{v})\,\frac{\partial\varphi}{\partial t} - [D(v)\nabla v - D(\tilde{v})\nabla\tilde{v}]\cdot\nabla\varphi \\ &- [v(A(v)\nabla u + B(v)) - \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))]\cdot\nabla\,\varphi\} = 0. \end{aligned} \tag{2.2}$$

Taking $\varphi = u - \tilde{u}$ in (2.1) and using the Cauchy inequality $2ab \le \epsilon a^2 + \frac{1}{\epsilon}b^2,\ \ \epsilon > 0$ we easily get

$$\begin{aligned} &\int_{\Omega\times\{t\}} |\nabla(u-\tilde{u})|^2 \le \\ &\le \frac{2}{A_0^2}\left(\max_{|s|\le M_1}|A'(s)|^2 \sup_Q |\nabla u|^2 + \max_{|s|\le M_1}|B'(s)|^2\right)\int_{\Omega\times\{t\}}|v-\tilde{v}|^2, \end{aligned} \tag{2.3}$$

where $M_1 = \max\{\sup_Q |v|, \sup_Q |\tilde{v}|\}$.
On the other hand, we can take $\varphi = v - \tilde{v}$ in (2.2) and obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\times\{t\}}|v-\tilde{v}|^2 + \int_0^t\!\!\int_\Omega [D(v)\nabla v - D(\tilde{v})\nabla\tilde{v}]\cdot\nabla(v-\tilde{v}) \\ &+ \int_0^t\!\!\int_\Omega [v(A(v)\nabla u + B(v)) - \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))]\cdot\nabla(v-\tilde{v}) = 0. \end{aligned} \tag{2.4}$$

Noting that

$$\begin{aligned} \int_0^t\!\!\int_\Omega (v-\tilde{v})(A(v)\nabla u + B(v))\cdot\nabla(v-\tilde{v}) &= \\ \frac{1}{2}\int_0^t\!\!\int_\Omega (A(v)\nabla u + B(v))\cdot\nabla(|v-\tilde{v}|^2) &= 0 \end{aligned}$$

we get from (2.4)

$$
\begin{aligned}
\int_{\Omega\times\{t\}}|v-\tilde v|^2 \;&+\; D_0\int_0^t\!\!\int_\Omega|\nabla(v-\tilde v)|^2\\
&\le \frac{4}{D_0}\int_0^t\!\!\int_\Omega\{\,[D(v)-D(\tilde v)]^2|\nabla v|^2+|\tilde v|^2|A(v)\nabla u-A(\tilde v)\nabla\tilde u|^2\\
&\quad +|\tilde v|^2|B(v)-B(\tilde v)|^2\,\}\\
&\le \frac{4}{D_0}\left[\max_{|s|\le M_1}|D'(s)|^2\sup_Q|\nabla v|^2+M_1^2\max_{|s|\le M_1}|A'(s)|^2\sup_Q|\nabla u|^2\right.\\
&\quad \left.+\,M_1^2\max_{|s|\le M_1}|B'(s)|^2\right]\int_0^t\!\!\int_\Omega|v-\tilde v|^2\\
&\quad +\frac{4}{D_0}M_1^2A_1^2\int_0^t\!\!\int_\Omega|\nabla(u-\tilde u)|^2. \qquad (2.5)
\end{aligned}
$$

Substituting (2.3) into (2.5) we arrive at

$$
\int_{\Omega\times\{t\}}|v-\tilde v|^2+D_0\int_0^t\!\!\int_\Omega|\nabla(v-\tilde v)|^2\le C_1\int_0^t\!\!\int_\Omega|v-\tilde v|^2, \qquad (2.6)
$$

where

$$
\begin{aligned}
C_1 \;&=\; \frac{4}{D_0}\left[\max_{|s|\le M_1}|D'(s)|^2\sup_Q|\nabla v|^2+M_1^2\left(1+\frac{2A_1^2}{A_0^2}\right)\max_{|s|\le M_1}|A'(s)|^2\sup_Q|\nabla u|^2\right]\\
&\quad +\frac{4M_1^2}{D_0}\left(1+\frac{2A_1^2}{A_0^2}\right)\max_{|s|\le M_1}|B'(s)|^2.
\end{aligned}
$$

Now a Gronwall argument applied to $\int_{\Omega\times\{t\}}|v-\tilde v|^2$ yields $v=\tilde v$, and then $u=\tilde u$ almost everywhere.

Remark. The condition $\nabla u,\nabla v\in L^\infty(Q)$ in the theorem is not restrictive. Lemmas 4.1 and 4.2 we will give later imply that any weak solution of problem (I) satisfies this condition.

3 Existence of weak solution

Lemma 3.1: *For a given $v\in L^\infty(Q)$ there exists a unique $u\in u_1+L^\infty(0,T;V)$ such that for any $\psi\in V$*

$$
\int_{\Omega\times\{t\}}[A(v)\nabla u+B(v)]\cdot\nabla\psi=0 \quad a.e.\quad t\in(0,T). \qquad (3.1)
$$

Moreover, we have for almost all $t \in (0,T)$

$$\int_{\Omega\times\{t\}} |\nabla u|^2 \;\leq\; C, \tag{3.2}$$

$$\| u(\cdot,t) \|_{C^\alpha(\bar\Omega)} \;\leq\; C, \tag{3.3}$$

where constants $C>0,\ \alpha\in(0,1)$ *depend only on known data and* $\sup_Q |v|$.

Proof: This is the standard result for an elliptic equation, cf.[5].

Lemma 3.2.: *For a given* $v \in L^\infty(Q)$ *let* $u \in u_1 + L^\infty(0,T;V)$ *be the weak solution of (3.1). Then there exists a unique* $\tilde v \in v_1 + L^2(0,T;V)\cap L^\infty(Q)$ *such that*

$$\iint_Q \left\{ \tilde v \frac{\partial\varphi}{\partial t} - [D(v)\nabla\tilde v + \tilde v(A(v)\nabla u + B(v))]\cdot\nabla\varphi \right\} + \int_{\Omega\times\{0\}} v_1\varphi = 0 \tag{3.4}$$

for any $\varphi \in C^1(\bar Q)$ *vanishing on* Σ_1 *and* $\Omega\times\{T\}$. *In addition,* $\tilde v$ *admits of the following estimates:*

$$\sup_Q |\tilde v| \;\leq\; \sup_{\Sigma_1\cup\Omega_0} |v_1|, \tag{3.5}$$

$$\iint_Q |\nabla\tilde v|^2 \;\leq\; C, \tag{3.6}$$

$$\| \tilde v \|_{C^\beta(\bar Q)} \;\leq\; C, \tag{3.7}$$

where constants $C>0,\ \beta\in(0,\alpha)$ *depend only on known data and* $\sup_Q |v|$.

Proof: We first modify equation (3.4): Let

$$p(s) = \min\left\{1, \frac{M}{|s|}\right\}\cdot s, \quad M>0$$

and consider

$$\iint_Q \left\{ \tilde v \frac{\partial\varphi}{\partial t} - [D(v)\nabla\tilde v + p(\tilde v)(A(v)\nabla u + B(v))]\cdot\nabla\varphi \right\} + \int_{\Omega\times\{0\}} v_1\varphi = 0. \tag{3.4a}$$

Clearly there exists a unique $\tilde v \in v_1 + L^2(0,T;V)\cap L^\infty$ such that (3.4a) holds for every $\varphi\in C^1(\bar Q)$ vanishing on Σ_1 and $\Omega\times\{T\}$.

Denote $\sup_{\Sigma_1 \cup \Omega_0} |v_1| = k$ and take $\varphi = (\tilde{v} - k)^+$ in (3.4a). Thus we get

$$\begin{aligned}\sup_{0 \le t \le T} \int_{\Omega \times \{t\}} |(\tilde{v} - k)^+|^2 + \iint_Q |\nabla (\tilde{v} - k)^+|^2 \\ \le C \left| \iint_Q p(\tilde{v})(A(v)\nabla u + B(v)) \cdot \nabla (\tilde{v} - k)^+ \right|.\end{aligned} \tag{3.8}$$

Set $\psi = \int_k^{\max(\tilde{v},k)} p(s) ds$.

Then for almost all $t \in (0, T)$,

$$\nabla \psi(\cdot, t) = p(\tilde{v}(\cdot, t)) \cdot \nabla (\tilde{v} - k)^+ \in L^2(\Omega).$$

So we can substitute this ψ into (3.1) and obtain

$$\int_{\Omega \times \{t\}} p(\tilde{v})(A(v)\nabla u + B(v)) \cdot \nabla (\tilde{v} - k)^+ = 0 \quad \text{a.e.} \quad t \in (0, T).$$

Hence (3.8) implies $\sup_Q \tilde{v} \le k$.
In the same way we obtain $\sup_Q (-\tilde{v}) \le k$.
Now by taking $M > k$ in (3.4a) we see that $p(\tilde{v}) = \tilde{v}$ in Q, which shows the equivalence of (3.4) and (3.4a). So far we have proved the existence of weak solutions for (3.4) and obtained estimate (3.5). Obviously (3.6) follows.
To prove (3.7) we first give some notations: For $z_0 = (x_0, t_0) \in Q$, $R \le R_0$ with $R_0 = \min \left\{ t_0^{\frac{1}{2}}, dist(x_0, \partial\Omega) \right\}$, set

$$\begin{aligned}
B_R(x_0) &= \{x \in \mathbb{R}^n : |x - x_0| \le R\}, \\
\Lambda_R(t_0) &= (t_0 - R^2, t_0), \\
Q_R(z_0) &= B_R(x_0) \times \Lambda_R(t_0), \\
u_{x_0,R}(t) &= \frac{1}{|B_R(x_0)|} \int_{B_R(x_0) \times \{t\}} u, \\
v_{z_0,R} &= \frac{1}{|Q_R(z_0)|} \int_{Q_R(z_0)} v.
\end{aligned}$$

We will omit z_0, x_0, t_0 henceforth.
By taking $\varphi = (u - u_{2R}(t))\eta^2$ in (3.1), where η is a cut-off function on B_{2R} with $2R \le R_0$ we get an inequality of the Cacciopolli type:

$$\int_{B_R \times \{t\}} |\nabla u|^2 \le \frac{C}{R^2} \int_{B_{2R} \times \{t\}} |u - u_{2R}(t)|^2 + CR^n.$$

Combining it with estimate (3.3) we then obtain

$$\int_{B_R\times\{t\}} |\nabla u|^2 \leq CR^{n-2+2\alpha} \quad \text{a.e.} \quad t\in(0,T). \tag{3.9}$$

In Q_R we write $\tilde{v} = V + W$ where $V \in L^2(\Lambda_R; H^1(B_R))$ satisfying $V_{|\partial Q_R} = \tilde{v}$ and

$$\iint_{Q_R}\left[V\frac{\partial\varphi}{\partial t} - D(v)\nabla V\cdot\nabla\varphi\right] = 0 \qquad \forall\varphi\in C_0^\infty(Q_R),$$

whereas $W = \tilde{v} - V \in L^2(\Lambda_R; H^1(B_R))$ with $W_{|\partial Q_R} = 0$ and

$$\iint_{Q}\left[W\frac{\partial\varphi}{\partial t} - D(v)\nabla W\cdot\nabla\varphi\right] = \iint_{Q_R}\tilde{v}[A(v)\nabla u + B(v)]\cdot\nabla\varphi$$

$\forall\varphi\in C_0^\infty(Q_R)$.
From the De Giorgi - Nask theorem we can get (see [8] for detail)

$$\iint_{Q_\rho}|V - V_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2+2\gamma}\iint_{Q_R}|V - V_R|^2, \qquad \forall\rho\leq R, \tag{3.10}$$

where constants $C>0,\ \gamma\in(0,1)$ depend only on D_0, D_1 and u.
It is known (see [8]) that $W\in W_2^{1,\frac{1}{2}}(Q_R)$ and

$$\| W \|^2_{W_2^{1,\frac{1}{2}}(Q_R)} \leq C\iint_{Q_R}|A(v)\nabla u + B(v)|^2. \tag{3.11}$$

On the other hand, we have (see [4])

$$\iint_{Q_R}|W - W_R|^2 \leq CR^2 \| W \|^2_{W_2^{1,\frac{1}{2}}(Q_R)}. \tag{3.12}$$

Putting (3.11) and (3.12) together and taking account of (3.9) we obtain

$$\iint_{Q_R}|W - W_R|^2 \leq CR^{n+2+2\alpha}, \tag{3.13}$$

where constant C depends only on known data and $\sup_Q|v|$.
Because $\tilde{v} = V + W$, from (3.10) and (3.13) it follows that

$$\iint_{Q_\rho}|\tilde{v} - \tilde{v}_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2+2\gamma}\iint_{Q_R}|\tilde{v} - \tilde{v}_R|^2 + CR^{n+2+2\alpha} \tag{3.14}$$

for any $\rho\leq R$ with $Q_{2R} < Q$. It is well known that (3.14) implies an interior C^β - estimate of $\tilde{v}$ where $0<\beta<\min\{\alpha,\gamma\}$.

We omit the estimation near the lateral boundary $\Sigma_1 \cup \Sigma_2$ and the bottom Σ_0. In fact, as u_1, v_1 and $\partial\Omega$ are of C^3 - class, the above argument with a trivial modification is valid for boundary estimation. The lemma is then proved.
Using the above two lemmas we can show

Theorem 3.1: *Problem (I) has at least one weak solution.*

Proof: Introduce the Banach space

$$X = L^2(0,T;H^1(\Omega)) \cap C^\gamma(\bar{Q})$$

equipped with the norm

$$\| \cdot \|_X = \| \cdot \|_{L^2(0,T;H^1(\Omega))} + \| \cdot \|_{C^\gamma(\bar{Q})},$$

where $\gamma \in (0,\beta)$ and β is the Hölder exponent in (3.7). Take a convex subset in X:

$$X_1 = \{w \in X : w_{|\Sigma_1 \cup \Omega_0} = v_1\}$$

and define $F : X_1 \to X_1$ as follows: For $v \in X_1$ let $F(v) = \tilde{v}$ which is given in Lemma 3.2. In virtue of Lemma 3.2, F is well defined and there is an $M > 0$ such that

$$F(X_1) \subset \{w \in X_1 : \| w \|_X \leq M\}.$$

Take $v_j \in X_1, j = 0, 1 \cdots$, satisfying $\| v_j - v_0 \|_X \to 0$ as $j \to \infty$.
Correspondingly, we get u_j and $\tilde{v_j} = F(v_j),\ j = 0, 1, \cdots$.
Just as the derivation of (2.3) and (2.6) we have

$$\begin{aligned}
\int_{\Omega\times\{t\}} |\nabla(u_j - u_0)|^2 &\leq C \sup_{\Omega\times\{t\}} |\tilde{v_j} - \tilde{v_0}|^2 \\
\sup_{0<t<T} \int_{\Omega\times\{t\}} |v_j - v_0|^2 &+ \iint_Q |\nabla(\tilde{v_j} - \tilde{v_0})|^2 \\
&\leq C \left[\iint_Q |\nabla(u_j - u_0)|^2 + \sup_Q |v_j - v_0|^2 \right].
\end{aligned}$$

Hence $\{\tilde{v_j}\}$ converges to $\tilde{v_j}$ in $L^2(0,T;H^1(\Omega))$ for $j \to \infty$.
On the other hand, from estimate (3.7) we see that $\{\tilde{v_j}\}$ is relatively compact in $C^\gamma(\bar{Q})$ and so in each subsequence of $\{\tilde{v_j}\}$ there exists a subsequence converging in $C^\gamma(\bar{Q})$. Accounting the convergence of $\{\tilde{v_j}\}$ in $L^2(0,T;H^1(\Omega))$ we then obtain

$$\| \tilde{v_j} - \tilde{v_0} \|_X \to 0 \quad \text{as} \quad j \to \infty,$$

which shows that $F : X_1 \to X_1$ is continuous. Analogously, we can verify the compactness of $F(X_1)$.
Now applying the Schauder fixed point theorem to F we obtain a function $v \in X_1$ such that $F(v) = v$. Furthermore, substituting v into (3.1) we get u. Then (u, v) is just a weak solution of problem (I). The proof is completed.

Remark. From the proofs of Lemmas 3.1 and 3.2 we see that there exist constants $C > 0, \alpha \in (0,1)$ such that for any weak solution (u, v) of problem (I),

$$\begin{aligned} \| u(\cdot,t) \|_{C^\alpha(\bar\Omega)} &\leq C \quad \text{a.e.} \quad t \in (0,T), \\ \| v \|_{C^\alpha(\bar\Omega)} &\leq C. \end{aligned}$$

4 Smoothness of weak solutions. Classical solution

From now on we suppose $\bar\Gamma_1 \cap \bar\Gamma_2 = \emptyset$.

Lemma 4.1: *Let (u, v) be a weak solution of (I). Then there exists an $\alpha \in (0,1)$ such that $\nabla u \in C^\alpha(\bar Q)$.*

Proof: According to the last remark we have

$$\begin{aligned} \| u(\cdot,t) \|_{C^\alpha(\bar\Omega)} &\leq C \quad \text{a.e.} \quad t \in (0,T), \\ \| v \|_{C^\alpha(\bar\Omega)} &\leq C. \end{aligned}$$

Thus in virtue of the $C^{1+\alpha}$ - regularity theory of elliptic equations [5] we arrive at $\nabla u(\cdot,t) \in C^\alpha(\bar\Omega)$ and

$$\| \nabla u(\cdot,t) \|_{C^\alpha(\bar\Omega)} \leq C \quad \text{a.e.} \quad t \in (0,T). \tag{4.1}$$

Now we turn to the continuity of u with respect to t. Define

$$\delta_h u(\cdot,t) = u(\cdot,t+h) - u(\cdot,t) \quad 0 < h < T,\ 0 \leq t \leq T-h.$$

It is easy to see that for any $\varphi \in V$

$$\int_{\Omega\times\{t\}} [A(v)\nabla\delta_h u + \delta_h B(v)]\nabla\varphi = -\int_\Omega \delta_h A(v(\cdot,t)) \cdot \nabla u(\cdot,t+h)\nabla\varphi \tag{4.2}$$

for $t \in [0, T-h]$. Regarding (4.2) as the equation for $\delta_h u$ and using L^∞ - estimates [5] we get

$$\begin{aligned} \sup_{\Omega\times\{t\}} |\delta_h u| &\leq C\left\{\sup_{\Omega\times\{t\}} |\delta_h u_1| + \sup_{\Omega\times\{t\}} |\delta_h A(v)| + \sup_{\Omega\times\{t\}} |\delta_h B(v)|\right\} \\ &\leq Ch^{\frac{\alpha}{2}} \quad \forall t \in [0, T-h],\ h \in [0, T]. \end{aligned} \tag{4.3}$$

This indicates the Hölder continuity of u with respect to t, with exponent $\frac{\alpha}{2}$. Now the lemma readily follows from an interpolation result given in [6]:

Proposition: *If $f(x,t) \in C(\bar{Q})$ satisfies*

$$\begin{aligned} \| f(\cdot, t) \|_{C^{m+\epsilon}(\bar{Q})} &\leq C_1 \qquad \forall t \in [0, T] \\ \| f(x, \cdot) \|_{C^{\lambda}([0,T])} &\leq C_2 \qquad \forall x \in \bar{\Omega} \end{aligned}$$

then $D_x^m f(x, \cdot) \in C^{\frac{\lambda\epsilon}{m+\epsilon}}([0, T])$ and

$$\| D_x^m f(x, \cdot) \|_{C^{\frac{\lambda\epsilon}{m+\epsilon}}([0,T])} \leq C_3,$$

where constant C_3 depends only on C_1, C_2.
The proof of Lemma 4.1 is then completed.

Lemma 4.2: *Let (u, v) be a weak solution of (I). Then there exists an $\alpha \in (0, 1)$ such that $\nabla v \in C^\alpha(\bar{Q})$.*

Proof: As in the proof of Lemma 3.2, we only give the interior estimate. We known that $v \in C^\alpha(\bar{Q})$. First we improve the Hölder exponent α. For any $Q_{2R} \subset Q$ split v in Q_{2R} as follows: $v = V + W$, where $V \in L^2(\Lambda_R, H^1(B_R))$, $V_{|\partial Q_R} = v$ and

$$\iint_{Q_R} \left[V\frac{\partial\varphi}{\partial t} - D(v_R)\nabla V \cdot \nabla\varphi\right] = 0 \qquad \forall\varphi \in C_0^\infty(Q_R),$$

whereas $W = v - V \in L^2(\Lambda_R, H^1(B_R))$, $W_{|\partial Q_R} = 0$ and

$$\iint_{Q_R} \left[W\frac{\partial\varphi}{\partial t} - D(v_R)\nabla W \cdot \nabla\varphi\right] = \iint_{Q_R} [(D(v) - D(v_R))\nabla v - vq] \cdot \nabla\varphi$$

for all $\varphi \in C_0^\infty(Q_R)$. It is known (see [4]) that

$$\iint_{Q_\rho} |v - v_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{n+4} \iint_{Q_R} |v - v_R|^2 \quad \forall \rho \leq R, \tag{4.4}$$

$$\begin{aligned} \iint_{Q_R} |W - W_R|^2 &\leq CR^2 \Big\{ \iint_{Q_R} |D(v) - D(v_R)|^2 |\nabla v|^2 \\ &\quad + \iint_{Q_R} |vq\nabla W| \Big\}. \end{aligned} \tag{4.5}$$

Taking account of

$$\iint_{Q_R} vq \cdot \nabla W = -\iint_{Q_R} Wq \cdot \nabla v$$

and $\sup_{Q_R} |W| \leq CR^\alpha$, $|q| \leq C$ we obtain

$$\left| \iint_{Q_R} vq \cdot \nabla W \right| \leq R^{2\alpha} \iint_{Q_R} |\nabla v|^2 + CR^{n+2}. \tag{4.6}$$

Substituting (4.6) into (4.5) and using the Cacciopolli inequality

$$\iint_{Q_R} |\nabla v|^2 \leq \frac{C}{R^2} \iint_{Q_{2R}} |v - v_{2R}|^2 + CR^{n+2}$$

which can be derived from the equation, we get

$$\iint_{Q_R} |W - W_R|^2 \leq CR^{2\alpha} \iint_{Q_{2R}} |v - v_{2R}|^2 + CR^{n+4}. \tag{4.7}$$

From (4.4) and (4.7) we see

$$\iint_{Q_\rho} |v - v_\rho|^2 \leq C\left[\left(\frac{\rho}{R}\right)^{n+4} + R^{2\alpha}\right] \iint_{Q_{2R}} |v - v_{2R}|^2 + CR^{n+4}. \tag{4.8}$$

Obviously (4.8) holds not only for $\rho \leq R$, but also for $R < \rho \leq 2R$, which implies the C^β-estimate of v, with arbitrary $\beta \in (0,1)$.
Now turn to the C^α-estimate for ∇v. Write $v = V + W$ as before, and then (see [4])

$$\iint_{Q_\rho} |\nabla V - (\nabla V)_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{n+4} \iint_{Q_R} |\nabla V - (\nabla V)_R|^2 \tag{4.9}$$

for all $\rho \leq R$,

$$\begin{aligned} \iint_{Q_R} |\nabla W|^2 &\leq \Big\{ \iint_{Q_R} |D(v) - D(v_R)|^2 |\nabla v|^2 \\ &\quad + \left| \iint_{Q_R} vq\nabla W \right| \Big\}. \end{aligned} \tag{4.10}$$

By the Cacciopolli inequality and the C^β-estimate of v we have

$$\iint_{Q_R} |D(v) - D(v_R)|^2 |\nabla v|^2 \le C R^{n+4\beta}. \tag{4.11}$$

Recalling the derivation of (4.6) we can get

$$\left| \iint_{Q_R} vq \cdot \nabla W \right| \le R \iint_{Q_R} |\nabla v|^2 + \frac{1}{4R} \iint_{Q_R} |W_q|^2 \le C R^{n+1+2\beta}. \tag{4.12}$$

From (4.9) - (4.12) it follows that

$$\iint_{Q_\rho} |\nabla v - (\nabla v)_\rho|^2 \le C \left(\frac{\rho}{R}\right)^{n+4} \iint_{Q_R} |\nabla v - (\nabla v)_R|^2 + C R^{n+2+2\gamma}$$

where $\gamma \in (0, \frac{1}{2})$.
Consequently, the C^α-continuity of ∇v follows and the lemma is proved.
From Lemmas 4.1 and 4.2 we immediately obtain the existence of classical solutions for problem (I). In fact, let (u, v) be a weak solution, hence $\nabla u, \nabla v \in C^\alpha(\bar{Q})$. Formally, we have

$$\begin{aligned} A(v)\Delta u + A'(v)\nabla v \cdot \nabla u + B'(v)\nabla v &= 0 \\ \frac{\partial v}{\partial t} + (q - D'(v)\nabla v) \cdot \nabla v - D(v)\Delta v &= 0. \end{aligned}$$

By the standard theory on linear equations we get

$$\begin{aligned} u(\cdot, t) &\in C^{2+\alpha}(\bar{\Omega}) \quad \forall t \in [0, T] \\ v &\in C^{2+\alpha}(Q) \cap C^{1+\alpha}(\bar{Q}). \end{aligned}$$

Combining these with Theorem 2.1 we then obtain:

Theorem 4.1: *Problem (I) has a unique classical solution (u, v), where*

$$\begin{aligned} u(\cdot, t) &\in C^{2+\alpha}(\bar{\Omega}) \quad \forall t \in [0, T] \\ v &\in C^{2+\alpha}(\tilde{Q}) \cap C^{1+\alpha}(\bar{Q}), \quad \tilde{Q} = \bar{Q} \setminus (\partial\Omega \times \{0\}); \end{aligned}$$

in addition, $u, \nabla u \in C^\alpha(\bar{Q})$.
Since we have set up the Hölder continuity of u in t and the uniform $C^{2+\alpha}$-estimate of $u(\cdot, t)$, we claim that the first two order x-derivatives of u are Hölder continuous in t. For the continuity of $\frac{\partial u}{\partial t}$ we have the following:

Theorem 4.2: *Let (u,v) be the solution of (I). Then $u \in C^{2+\alpha}(\bar{\Omega} \times (0,T])$ with some $\alpha \in (0,1)$.*

Proof: Recalling (4.2) as an equation of $\delta_h u$ we have

$$\int_{\Omega\times\{t\}} |\nabla \partial_t^h u|^2 \leq C \int_{\Omega\times\{t\}} \left(|\partial_t^h B(v)|^2 + |\partial_t^h A(v)|^2 + |\nabla \partial_t^h u_1|^2\right) \leq C(\tau)$$

for $\tau \leq t \leq T-h$ with $0 < \tau < T-h$, where

$$\partial_t^h u(\cdot,t) = \frac{1}{h}\delta_h u(\cdot,t).$$

By the L^∞-estimate we get

$$\sup_\Omega |\partial_t^h u(\cdot,t)| \leq C(\tau), \quad \forall 0 < \tau \leq t \leq T-h,$$

and so

$$\frac{\partial u}{\partial t} \in L^\infty(\tau,T;H^1(\Omega)) \cap L^\infty(\Omega\times(\tau,T))$$

for any $\tau > 0$. Moreover, we have

$$\int_{\Omega\times\{t\}} A(v)\nabla\frac{\partial u}{\partial t}\cdot\nabla\varphi = -\int_{\Omega\times\{t\}}\left[A'(v)\frac{\partial v}{\partial t}\nabla u + B'(v)\frac{\partial v}{\partial t}\right]\cdot\nabla\varphi,$$

$\forall \varphi \in H_0^1(\Omega),\ t \in (0,T]$. Now applying the C^α-estimate to $\frac{\partial u}{\partial t}$ we obtain

$$\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{C^\alpha(\bar{\Omega})} \leq C(\tau), \quad 0 < \tau \leq t \leq T. \tag{4.13}$$

As in the proof of Lemma 4.1, we find

$$\sup_\Omega \left|\delta_h \frac{\partial u}{\partial t}(\cdot,t)\right| \leq C(\tau)h^{\frac{\alpha}{2}}, \quad 0 < \tau \leq t \leq T. \tag{4.14}$$

From (4.13) - (4.14) it follows that

$$\frac{\partial u}{\partial t} \in C^\alpha(\bar{\Omega}\times[\tau,T])$$

for any $\tau > 0$, thus completing the proof.

Corollary 4.1: *If all known data are C^∞-smooth, then the solution (u,v) of (I) is also C^∞-smooth. To be more precise, we have*

$$\begin{aligned} D_x^m u &\in C(\bar{Q}), \quad m \geq 0; \\ D_x^m D_t^k u &\in C(\bar{\Omega} \times (0,T]), \quad m \geq 0, \ k \geq 1; \\ D_x^m v &\in C^\alpha(\bar{Q}), \quad m = 0,1; \\ D_x^m D_t^k v &\in C(\tilde{Q}), \quad m \geq 0, \ k \geq 0. \end{aligned}$$

Note that in both Theorems 4.1, 4.2 and Corollary 4.1, the smoothness of higher order for u and v is obtained only in $\bar{\Omega} \times (0,T]$ or $\tilde{Q}$, but not on the whole $\bar{Q}$. As a matter of fact, it is easy to see from the proof of Theorem 4.2 that the continuity of $\frac{\partial u}{\partial t}$ depends on the continuity of $D_x^2 v$ and $\frac{\partial v}{\partial t}$. On the other hand, $v \in C^{2+\alpha}(\bar{Q})$ if and only if the following compatibility condition is satisfied:

$$\frac{\partial v_1}{\partial t} - \nabla[D(v_1)\nabla v_1] - A(v_1)\nabla \hat{u} + B(v_1)] \cdot \nabla v_1 = 0 \quad \text{at} \quad \Gamma_1 \times \{0\}, \tag{4.15}$$

where $\hat{u}$ is determined by the following boundary value problem

$$\begin{cases} \nabla[A(v_1(\cdot,0))\nabla \hat{u} + B(v_1(\cdot,0))] = 0 & \text{in} \quad \Omega \\ \hat{u}_{|\Gamma_1} = u_1(\cdot,0), \quad [A(v_1(\cdot,0))\nabla \hat{u} + B(v_1(\cdot,0))] \cdot \nu_{|\Gamma_2} = 0. \end{cases} \tag{4.16}$$

Therefore we obtain

Corollary 4.2: *Let (u,v) be the solution of (I) with (u_1,v_1) satisfying (4.15) and (4.16). Then $u,v \in C^{2+\alpha}(\bar{Q})$.*

5 Large time behavior

To start with, we define the ω-limit set of problem (I) as

$$\begin{aligned} \omega = \{(\, w_1, w_2) : \quad & \text{there exist a subsequence} \quad \{t_k\} \\ & \text{such that} \quad t_k \to \infty \text{ and } \lim_{t_k \to \infty} u(x,t_k) = w_1(x), \\ & \lim_{t_k \to \infty} v(x,t_k) = w_2(x)\,\}\,, \end{aligned}$$

where (u,v) is the solution of (I).
Since the regularity estimates on $\bar{\Omega} \times [\tau, T]$ are independent of T, $\{(u(\cdot,t), v(\cdot,t))\}$ is relatively compact in $[C^{2+\alpha}(\bar{\Omega})]^2$, where t is regarded as a parameter. Hence, $\omega \subset [C^{2+\alpha}(\bar{\Omega})]^2$ is non-empty.

In this section we assume

$$\begin{aligned}\lim_{t\to\infty} u_1(x,t) &= u_0(x)\\ \lim_{t\to\infty} v_1(x,t) &= v_0(x)\end{aligned}$$

uniformly for $x \in \Gamma_1$, and then we have

Theorem 5.1: *All of the functions in ω are the steady solutions of problem (I), i.e. for any $(\tilde{u},\tilde{v}) \in \omega$ we have*

$$(I_0)\quad \begin{cases} \nabla[A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}] = 0 & in \quad \Omega \\ \nabla[D(\tilde{v})\nabla\tilde{v} + \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))] = 0 & in \quad \Omega \\ \tilde{u}_{|\Gamma_1} = u_0, \quad [A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}] \cdot \nu_{|\Gamma_2} = 0 & \\ \tilde{v}_{|\Gamma_1} = v_0, \quad [D(\tilde{v})\nabla\tilde{v} + \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))] \cdot \nu_{|\Gamma_2} = 0 & . \end{cases}$$

Proof: Let $(\tilde{u},\tilde{v}) \in \omega$ such that

$$u(\cdot,t_k) \to \tilde{u}, \quad v(\cdot,t_k) \to \tilde{v}$$

in the sense of $C^{2+\alpha}(\bar{Q})$ as $t_k \to \infty$. From now on we simply write

$$u(t) = u(\cdot,t), \quad v(t) = v(\cdot,t).$$

Thus we have

$$\nabla \cdot [A(v(t))\nabla u(t) + B(v(t))] = 0 \tag{5.1}$$

$$\nabla \cdot [A(v(t_k))\nabla u(t_k) + B(v(t_k))] = 0. \tag{5.2}$$

Subtracting (5.2) from (5.1), multiplying by $u(t) - u(t_k)$, and integrating over Ω we get

$$\int_\Omega |\nabla(u(t) - u(t_k))|^2 \le C\left\{\int_\Omega |v(t) - v(t_k)|^2 + \int_{\Gamma_1} |u(t) - u(t_k)|\right\}. \tag{5.3}$$

We also get in a similar way

$$\begin{aligned}&\int_\Omega |v(\tau) - v(t_k)|^2 + \int_{t_k}^{\tau}\int_\Omega |\nabla(v(t) - v(t_k))|^2 \\ &\le C\left\{\int_{t_k}^{\tau}\int_\Omega |v(t) - v(t_k)|^2 + \int_{t_k}^{\tau}\int_\Omega |\nabla(u(t) - u(t_k))|^2 + \int_{t_k}^{\tau}\int_{\Gamma_1} |v(t) - v(t_k)|\right\}\end{aligned} \tag{5.4}$$

where $\tau > t_k$. From (5.3) and (5.4) it follows that

$$\int_\Omega |v(\tau) - v(t_k)|^2 + \int_{t_k}^{\tau}\int_\Omega |\nabla(v(t) - v(t_k))|^2$$
$$\leq C\left\{\int_{t_k}^{\tau}\int_\Omega |v(t) - v(t_k)|^2 + \int_{t_k}^{\tau}\int_{\Gamma_1} (|u(t) - u(t_k)| + |v(t) - v(t_k)|)\right\}. \quad (5.5)$$

Applying the Gronwall inequality yields

$$\int_\Omega |v(t) - v(t_k)|^2 \leq Ce^{c(t-t_k)} \int_{t_k}^{\tau}\int_{\Gamma_1} (|u(t) - u(t_k)| + |v(t) - v(t_k)|) \quad (5.6)$$

for $t_k \leq t \leq \tau$. Fix $\tau = t_k + 2$ and then from (5.6) we see that for any $t_k \leq t \leq t_k + 2$

$$v(t) \to \tilde{v} \quad \text{as} \quad t_k \to \infty.$$

Furthermore, from (5.3) and (5.5) we find

$$\nabla u(t) \to \nabla\tilde{u}, \quad \nabla v(t) \to \nabla\tilde{v}.$$

Now for any $\varphi \in V$ and $\rho(t) \in C_0^1((0,2))$ satisfying

$$\int_0^2 \rho(t) = 1, \quad \int_0^2 \rho'(t) = 0$$

we have

$$\begin{aligned}
\int_{t_k}^{t_k+2}\int_\Omega \frac{\partial v(t)}{\partial t}\varphi\rho(t - t_k) &= -\int_{t_k}^{t_k+2}\int_\Omega \varphi v(t)\rho'(t - t_k) \\
&= -\int_0^2\int_\Omega \rho'(s)\varphi v(t_k + s) \\
&\to -\int_0^2 \rho'(s)\int_\Omega \varphi\tilde{v} = 0 \quad \text{as} \quad t_k \to \infty.
\end{aligned}$$

On the other hand,

$$\int_{t_k}^{t_k+2}\int_\Omega \nabla\cdot[v(t)q(t) - D(v(t))\nabla v(t)]\varphi\rho(t - t_k)$$
$$= -\int_0^2 \rho(s)\int_\Omega [v(t_k + s)q(t_k + s) - D(v(t_k + s))\nabla v(t_k + s)]\cdot\nabla\varphi$$
$$\to -\int_\Omega [\tilde{v}\tilde{q} - D(\tilde{v})\nabla\tilde{v}]\cdot\varphi \quad \text{as} \quad t_k \to \infty.$$

Thus we arrive at

$$\int_\Omega [D(\tilde{v})\nabla\tilde{v} + \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))]\cdot\nabla\varphi = 0 \quad \forall\varphi\in V.$$

From this it immediately follows that

$$\nabla[D(\tilde{v})\nabla\tilde{v} + \tilde{v}(A(\tilde{v})\nabla\tilde{u} + B(\tilde{v}))] = 0 \quad \text{in} \quad \Omega.$$

Finally, letting $t_k \to \infty$ in (5.2), we then obtain

$$\nabla\cdot[A(\tilde{v})\nabla\tilde{u} + B(\tilde{v})] = 0.$$

As for the boundary conditions, they are obviously satisfied. Thus the theorem is proved.

By imposing more conditions on the structure of the system or the boundary data, we can prove

Theorem 5.2: *Assume that one of the following conditions is satisfied*

(i) : $v_0 = const$, *or*

(ii) : $A(v) = A_0,\ D(v) = D_0$ *and*

$$C_0|\Omega|^{\frac{1}{n}}(M-m)\max_{m\le s\le M}|B'(s)|D_0^{-1} < 1,$$

where C_0 *is an absolute constant,* $M = \max_{\Gamma_1} v_0,\ m = \min_{\Gamma_1} v_0$. *Then the solution of problem* (I_0) *is unique, and consequently, the solution of (I) converges to the solution of* (I_0) *in the sense of* $C^{2+\alpha}(\bar{\Omega})$ *as* $t\to\infty$.

Proof: If condition (ii) is satisfied, it needs only to check the proof of Theorem 2.1. If condition (i) is satisfied, constant v_0 is the unique solution to

$$\begin{cases} \nabla[D(\tilde{v})\nabla\tilde{v}] + [A(\tilde{v})\nabla\tilde{u} + B(\tilde{v})]\cdot\nabla\tilde{v} = 0 \\ \tilde{v}_{|\Gamma_1} = v_0, \quad \nabla\tilde{v}\cdot\nu_{|\Gamma_2} = 0 \end{cases}$$

and thus the solution to

$$\begin{cases} \nabla\cdot[A(v_0)\nabla\tilde{u}] + B(v_0)] = 0 \\ \tilde{u}_{|\Gamma_1} = u_0, \quad [A(v_0)\nabla\tilde{u} + B(v_0)]\cdot\nu_{|\Gamma_2} = 0 \end{cases}$$

is unique.

Remark: In general, condition (ii) in Theorem 5.2 is necessary for the uniqueness of problem (I_0). In the case of thermo-convection in porous media this condition is called the small Rayleigh number condition (cf. [3]).

References

[1] Alt, H. W., DiBenedetto, E., Nonsteady flow of water and oil through inhomogeneous porous media, *Ann. Scula Norm. Sup.*, Pisa, **12** (1985), 335–392.

[2] Bear, J., Dynamics of Fluids in Porous Media, *Amer. Elsev.,* New York, 1972.

[3] Caltagirone, J. P., Thermoconvective instabilities in a horizontal porous layer, *J. Fluid Mech.,* **72** (1975), 268–287.

[4] Campanato, S., Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth, *Rend. Sem. Mat. Univ. Padova,* **64** (1981), 59–75.

[5] Gilbarg, D., Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, 2nd Ed., *Springer,* New York, 1983.

[6] Kruzhkov, S. N., Kastro, A. Lopes, M., Estimates of Schauder type and existence theorems of solutions of basic problems for linear and nonlinear parabolic equations, *Dokl. Akad. Nauk SSSR*, **220** (1975), 277–280.

[7] Kruzhkov, S. N., Sukorjanski, S. M., Boundary value problems for systems of equation of two phase porous flow type: statement of the problems, questions of solvability, justification of approximate methods, *Mat. Sb.*, **107** (1977), 69–88.

[8] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'tseva, N. N., Linear and Quasilinear Equations of Parabolic Type, *Nauka,* Moscow, 1967.